Linear Programming Models

Here we will be investigating examples of special linear programming models known as

Network Flow Models

The six examples will be

(1) transportation

(2) transshipment

(3) assignment

(4) maximal-flow

(5) shortest-path

(6) minimal-spanning tree models.

1. Transportation Model – This model is concerned with the distributing of products from their suppliers (known as origins or sources) to their destinations (or sinks). The most common objective of the Transportation Model is to minimize transportation costs.

2. Transshipment Model – This is a model similar to the Transportation Model, the difference being that in a Transshipment Model, some of the points can have shipments that arrive as well as shipments that leave. In this scenario, companies may be able to save money by having goods from different sources shipped to a central location and then sending them out together to their destinations. One example of a Transshipment Model is the hub-and-spoke system.

3. Assignment Model – This model attempts to find the most advantageous assignments of people to tasks, jobs to machines, etc. The main objective is usually to either minimize cost or minimize time.

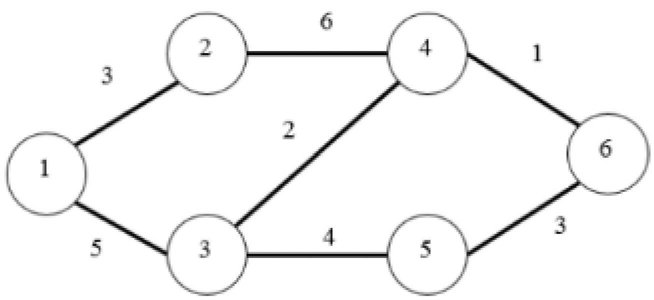
4. Maximal-flow Model – This model is designed to find the maximum flow that can occur through a network per hour/day/year.

5. Shortest-Path Model – This model is used to find the shortest route from the source to the sink (destination). This is similar to how a GPS can find the shortest route from one city to another using the network of existing roadways.

6. Minimal-Spanning Tree Models – This model connects all of the nodes in a network. The objective is usually to minimize the distance of the path that connects all of the nodes.

Network:

A network is comprised of nodes and arcs. Arcs are the lines that connect the nodes together.



This is an example of a network where the circles are the nodes and the straight lines are the arcs.

Examples of networks:

- Nodes: cities Arcs: roads that connect the cities
- Nodes: Houses in a neighborhood Arcs: Water pipes that connect the homes
- Nodes: Organs in a body Arcs: Blood vessels that connect the organs

Arcs:

Arcs can be either unidirectional or bidirectional. An unidirectional arc allows flow in only one direction while a bidirectional arc allows flow in both directions. In a diagram, a bidirectional arc is often represented by a pair of unidirectional arcs with opposite flow directions.

Arcs can also be either capacitated or uncapacitated. If an arc is capacitated, it has a limited capacity for flow, an uncapacitated arc does not have a flow constraint that will effect the model.

Nodes:

There are three basic types of nodes: supply, demand and transshipment.

A supply node is a site that is an origin or source such as a factory that manufactures products. This is where the goods enter the network.

Demand nodes specify a location that goods are delivered. These nodes, also known as sinks or destinations, denote the final stop for the product in the network.

A transshipment node is place where goods pass through on their way to or from a location. The node can serve as a redirection point. An example would be goods leaving a factory (supply node) and are transported to three different distribution sites (transshipment nodes). Here they will be collected and sorted and ultimately sent to many stores (demand nodes).

Network characteristics:

- In each network model, the variables that we are concerned with, known as the decision variables, represent the amount of flow that occurs in an unidirectional arc.
- Each node in the network will have an equation to calculate its net flow. These equations are known as flow balance equations:

  $$Net\ Flow = Total\ Flow\ into\ Node - Total\ Flow\ out\ of\ the\ Node$$

  If you think about the nature of the different type of nodes, you will realize that the total flow out of a supply node will exceed the total flow into the nodes so the net flow will be negative.
  In a demand node will have more flow coming in than going out since this is a destination. Therefore, the net flow at a demand node will be positive.
  Since the transshipment node serves as a redistribution center, the flow into these nodes will be the same as the flow out. So, the net flow for a transshipment node should be zero.
- In the constraint equations, the number in front of the decision variables (known as constraint coefficient) is always either 0 or 1. It is 0 when that decision variable does not exist in the constraint of the model of the network. And it is 1 when the variable does exist in the model.
- If all of the supplies and demands in network are integer values, then all flows will have an integer value.

Assignment Model

We discussed that the assignment model's objective is to identify an optimal one-to-one assignment of jobs to machines, people to tasks or product to transportation with minimal cost. To display an assignment model, we create a table. In this table, the rows are for the people or jobs we need to assign and the columns are for the tasks or machines to which we want them assigned. The numbers in the table generally represent the costs associated with each one-to-one assignment.

Example:

To illustrate the assignment model, consider Wickham Glass company. The employ four workers and have been given four jobs: a windshield, storefront, residential window and insulated glass. The cost for each of the four workers performing these jobs will differ because of the worker's speed and skill.

We want to assign these tasks to the workers in a way that will result in the lowest total cost. Each task must have one worker and each worker must have one task.

Here are the costs for each of the employees completing each of the jobs:

	Windshield	Storefront	Residential	Insulated
Employee 1	$25	$30	$18	$20
Employee 2	$18	$20	$24	$28
Employee 3	$21	$19	$22	$24
Employee 4	$29	$32	$25	$18

Since this is small case with only 4 assignments, we can make a chart with each of the 24 possibilities and see which one would minimize the cost. After we do that, we will look at a more general way to solve these questions

Employee-Project Assignment					
Windshield	Storefront	Residential	Insulated	Labor Costs	Total Cost
1	2	3	4	$25 $20 $22 $18	$85
1	2	4	3	$25 $20 $25 $24	$94
1	3	2	4	$25$19 $24 $18	$86
1	3	4	2	$25 $19 $25 $28	$97
1	4	2	3	$25 $32 $24 $24	$105
1	4	3	2	$25 $32 $22 $28	$107
2	1	3	4	$18 $30 $22 $18	$88
2	1	4	3	$18 $30 $25 $24	$97
2	3	1	4	$18 $19 $18 $18	$73
2	3	4	1	$18 $19 $25 $20	$82
2	4	1	3	$18 $32 $18 $24	$92
2	4	3	1	$18 $32 $22 $20	$92
3	1	2	4	$21 $30 $24 $18	$93
3	1	4	2	$21 $30 $25 $28	$104
3	2	1	4	$21 $20 $18 $18	$77

3	2	4	1	$21 $20 $25 $20	$86
3	4	1	2	$21 $32 $18 $28	$99
3	4	2	1	$21 $32 $24 $20	$97
4	1	2	3	$29 $30 $24 $24	$107
4	1	3	2	$29 $30 $22 $28	$109
4	2	1	3	$29 $20 $18 $24	$91
4	2	3	1	$29 $20 $22 $20	$91
4	3	1	2	$29 $19 $18 $28	$94
4	3	2	1	$29 $19 $24 $20	$92

The assignment that will cost the least is:

Windshield – Employee 2

Storefront – Employee 3

Residential – Employee 1

Insulated – Employee 4

For a total cost of $73

There are several methods for solving these types of problems. When the number of possibilities are relatively small, as is the case in the above example, it is feasible to list each possibility and determine the option with the least cost. However, when the number of possibilities increases, we must use a different method for determining the configuration with the least cost.

Assignment Model Solution:

We define X_{ij} as:

$$i - denotes\ worker$$

$$j - denotes\ task$$

So X_{ij} is the conduit between worker i and task j. The value will be 1 if that worker is assigned to that task and zero is the worker is not assigned to that task.

So, $i = 1(employee\ 1), 2(employee\ 2), 3(employee\ 3), 4(employee\ 4)$

$$j = W(windshield), S(storefront), R(residential), I(insulated)$$

Now we will determine the objective function and the constraint functions.

The objective function is the one that we wish to minimize, that is the total cost:

$$Total\ costs = \$25X_{1W} + \$30X_{1S} + \$18X_{1R} + \$20X_{1I} + \$18X_{2W} + \$20X_{2S} + \$24X_{2R} + \$28X_{2I} + \$21X_{3W} + \$19X_{3S} + \$22X_{3R} + \$24X_{3I} + \$29X_{4W} + \$32X_{4S} + \$25X_{4R} + \$18X_{4I}$$

Then we need to create the supply and demand constraints:

Supply constraints:

$$-X_{1W} - X_{1S} - X_{1R} - X_{1I} = -1$$

$$-X_{2W} - X_{2S} - X_{2R} - X_{2I} = -1$$

$$-X_{3W} - X_{3S} - X_{3R} - X_{3I} = -1$$

$$-X_{4W} - X_{4S} - X_{4R} - X_{4I} = -1$$

Demand constraints:

$$X_{1W} + X_{2W} + X_{3W} + X_{4W} = 1$$

$$X_{1S} + X_{2S} + X_{3S} + X_{4S} = 1$$

$$X_{1R} + X_{2R} + X_{3R} + X_{4R} = 1$$

$$X_{1I} + X_{2I} + X_{3I} + X_{4I} = 1$$

So, we need to set up the problem in Microsoft Excel. We begin by making a table with Employees listed down the first column and the tasks listed across the first row. Then the table is completed using dollar amount of each employee/task combination.

Cost	Windshield	Storefront	Residential	Insulated
Employee 1	$25	$30	$18	$20
Employee 2	$18	$20	$24	$28
Employee 3	$21	$19	$22	$24
Employee 4	$29	$32	$25	$18

Then make an identical table with the cost deleted. This will be our binary decision variable with 1 representing yes and 0 for no. Then, outside the table, we will input the formula to sum each of the rows and each of the columns. You can see these in row 15 and column G. Each of these sums must be equal to one since each employee should have one job and each job should have one employee.

	A	B	C	D	E	F	G	H	I
1									
2	Cost	Windshield	Storefront	Residential	Insulated				
3	Employee 1	$25	$30	$18	$20				
4	Employee 2	$18	$20	$24	$28				
5	Employee 3	$21	$19	$22	$24				
6	Employee 4	$29	$32	$25	$18				
7									
8									
9	Assignment	Windshield	Storefront	Residential	Insulated		Task Assigned		Supply
10	Employee 1	0	0	0	0		0	=	1
11	Employee 2	0	0	0	0		0	=	1
12	Employee 3	0	0	0	0		0	=	1
13	Employee 4	0	0	0	0		0	=	1
14									
15	Persons Assigned	0	0	0	0				
16		=	=	=	=				Total Cost
17	Demand	1	1	1	1				

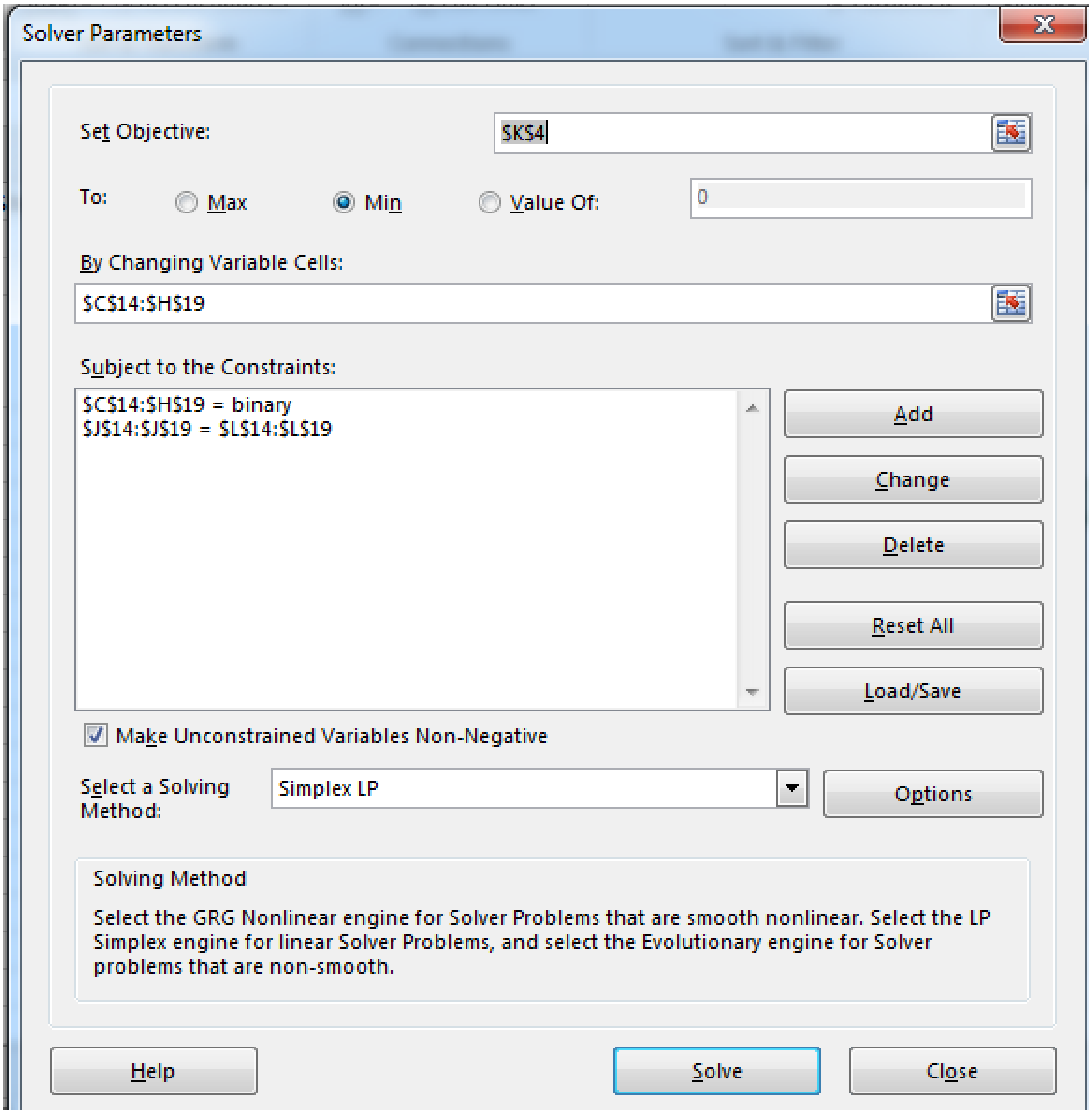
	A	B	C	D	E	F	G
1							
2	Cost	Windshield	Storefront	Residential	Insulated		
3	Employee 1	25	30	18	20		
4	Employee 2	18	20	24	28		
5	Employee 3	21	19	22	24		
6	Employee 4	29	32	25	18		
7							
8							
9	Assignment	Windshield	Storefront	Residential	Insulated		Task Assigned
10	Employee 1	0	0	0	0		=SUM(B10:E10)
11	Employee 2	0	0	0	0		=SUM(B11:E11)
12	Employee 3	0	0	0	0		=SUM(B12:E12)
13	Employee 4	0	0	0	0		=SUM(B13:E13)
14							
15	Persons Assigned	=SUM(B10:B13)	=SUM(C10:C13)	=SUM(D10:D13)	=SUM(E10:E13)		
16		=	=	=	=		
17	Demand	1	1	1	1		

The next thing we need to do is to define the Objective Function. This is the total cost which is the ‘sumproduct’ of table 1 and table two. We enter this formula into one of the cells in Excel

Task Assigned		Supply
=SUM(B10:E10)	=	1
=SUM(B11:E11)	=	1
=SUM(B12:E12)	=	1
=SUM(B13:E13)	=	1
		Total Cost
		=SUMPRODUCT(B3:E6,B10:E13)

Next we will use the Solver function of Microsoft Excel. If yours does not have the Solver Function, please click on https://support.office.com/en-us/article/load-the-solver-add-in-in-excel-612926fc-d53b-46b4-872c-e24772f078ca for directions on how to upload this add-in.

Click on the objective cell and then click on the data tab and then on Solver. Make sure the solver is set to the minimum value and then input the second table as the variable cells. We then need to add the restraints, one at a time, to the solver. First, set the variable cells to be binary. Then set all of the row and column total equal to 1 and set the solving system to Simplex LP



Click solve and then okay and you will get your results:

8									
9	Assignment	Windshield	Storefront	Residential	Insulated		Task Assigned		Supply
10	Employee 1	0	0	1	0		1	=	1
11	Employee 2	1	0	0	0		1	=	1
12	Employee 3	0	1	0	0		1	=	1
13	Employee 4	0	0	0	1		1	=	1
14									
15	Persons Assigned	1	1	1	1				
16		=	=	=	=				Total Cost
17	Demand	1	1	1	1				73
18									

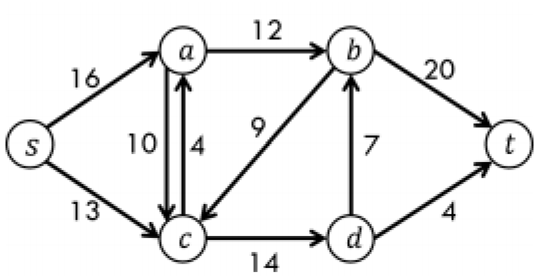
This shows the same results that we found by hand.

For more information on Assessment models, visit the website: https://www.youtube.com/watch?v=APTweXuMw3w

Maximal Flow Model

We use a maximal flow model to help us determine the maximum amount that can flow from an origin node to a destination node in a network with arcs that are capacitated. In other words, we are trying to maximize the flow on the arcs.

Below we have a map of the road system showing routes from Point s to Point t. We want to determine the maximum number of cars that can flow from *s* to *t.*



The number on the arcs represent the number of cars (in hundreds of cars per hour) that can flow from one node to another. As you can see traffic can flow both ways between two nodes, for example between a and c, and in only one direction between other nodes, such as a and b.

So there are the following paths:

From	To	Capacity per hour

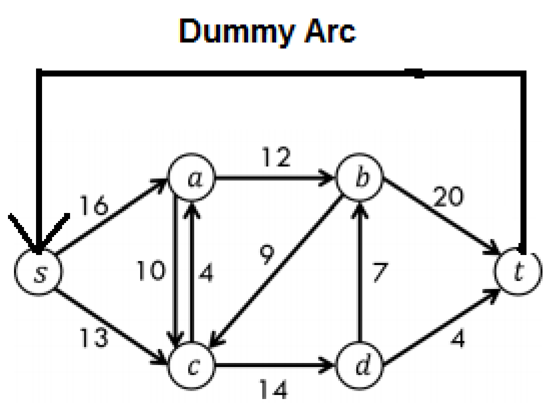
s	a	1600
s	c	1300
a	b	1200
a	c	1000
b	c	2000
b	t	900
c	a	400
c	d	1400
d	b	700
d	t	400

There are 10 different arcs, so there are 10 decision variables one for each arc.

Let X_{ij} be the number of cars that travel in an hour from node i to node j.

We can see that node s is the origin and node t is the destination and all other nodes are transshipment nodes since cars neither begin nor end there. Since there is not a set number of cars available at the beginning nor a set number of demand for cars at the end, we need to make slight modifications within the network. This modification is the addition of a unidirectional arc from the destination to the origin. This arc is known as a dummy arc, since there is no actual road present and we also set the capacity of this arc as infinity or some unreasonably high number.

The objective here is to maximize the number of vehicles that arrive at node t per hour, which is the same at the number of vehicles that depart node s per hour. Since there is not a supply at t, nor a demand at s, the number of cars that flow out of t must be the same as the number that flow into t, So we can think of the objective as maximizing the flow along the dummy arc going from t to s.



So the objective can be written as:

$$Maximize\ X_{ts}$$

By definition, the net flow in the transshipment nodes is zero, so we can use this fact to create constraints:

$(X_{sa} + X_{ca}) - (X_{ac} + X_{ab})$	= 0	Net flow at a
$(X_{ab} + X_{db}) - (X_{bt} + X_{bc})$	= 0	Net flow at b
$(X_{sc} + X_{ac} + X_{bc}) - (X_{cd} + X_{ca})$	= 0	Net flow at c
$X_{cd} - (X_{db} + X_{dt})$	= 0	Net flow at d

We can also state the capacity constraints:

$$X_{sa} \leq 16$$

$$X_{sc} \leq 13$$

$$X_{ab} \leq 12$$

$$X_{ac} \leq 10$$

$$X_{bc} \leq 20$$

$$X_{bt} \leq 9$$

$$X_{ca} \leq 4$$

$$X_{cd} \leq 14$$

$$X_{db} \leq 7$$

$$X_{dt} \leq 4$$

Next, we will use the Excel solver function to find the maximum flow from s to t. We begin by setting up a table with all of the maximum flows from one node to another. Notice that for our dummy variable, we need to put a large number there to make sure that we do not restrict that number.

	A	B	C	D	E	F	G	H	I	J
1										
2										
3				Node S	Node A	Node B	Node C	Node D	Node T	
4			Node S	0	16	0	13	0	0	
5			Node A	0	0	12	10	0	0	
6			Node B	0	0	0	9	0	20	
7			Node C	0	4	0	0	14	0	
8			Node D	0	0	7	0	0	4	
9			Node T	1000000	0	0	0	0	0	

Next we make a table for our decision variables. The last column adds all of the numbers in that row to give us the total flow out of the node and the last row adds all of the numbers in that column to give us the total flow into that node.

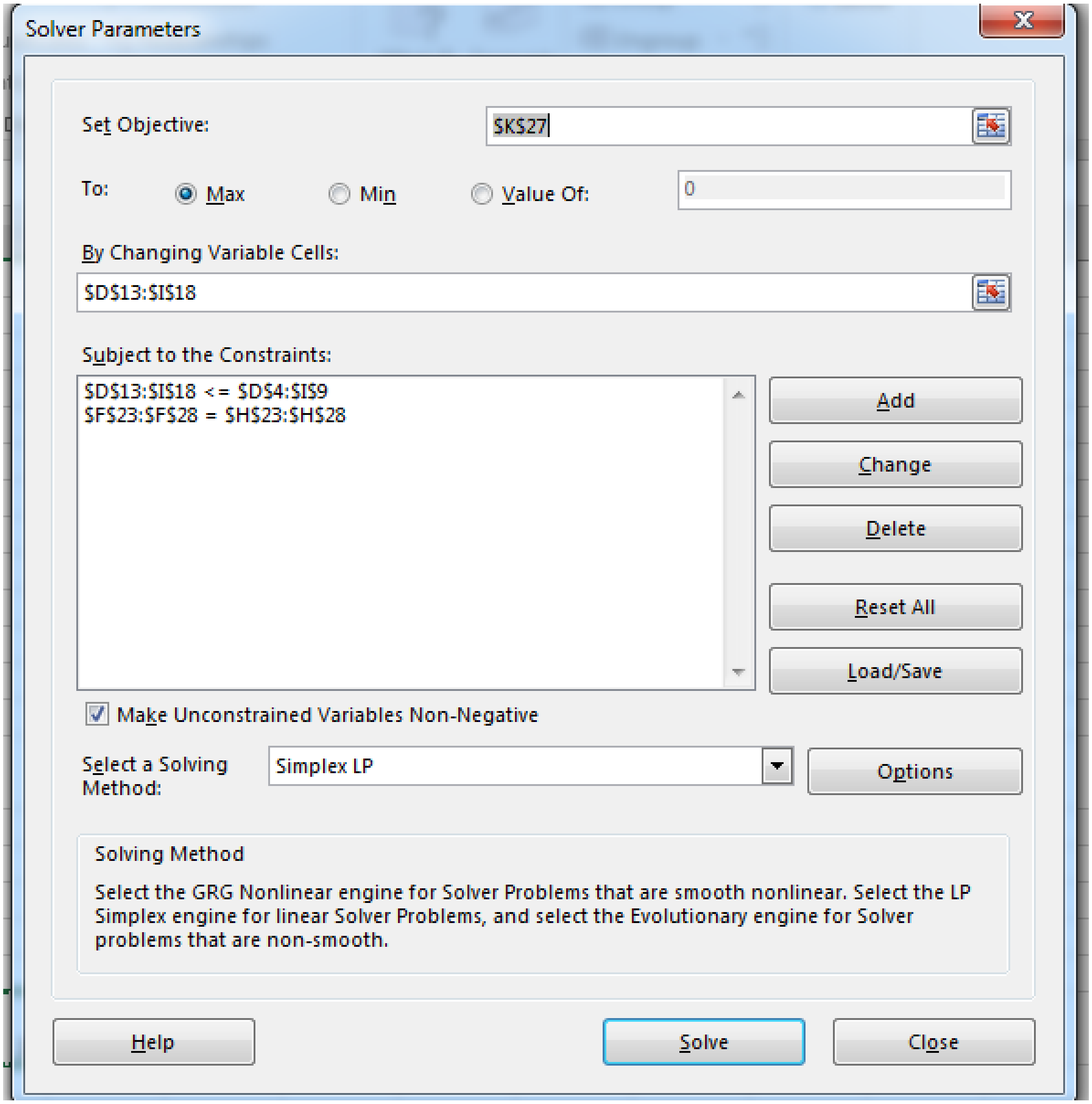
								Total Out
		Node S	Node A	Node B	Node C	Node D	Node T	
	Node S							=SUM(D13:I13)
	Node A							=SUM(D14:I14)
	Node B							=SUM(D15:I15)
	Node C							=SUM(D16:I16)
	Node D							=SUM(D17:I17)
	Node T							=SUM(D18:I18)
Total In		=SUM(D13:D18)	=SUM(E13:E18)	=SUM(F13:F18)	=SUM(G13:G18)	=SUM(H13:H18)	=SUM(I13:I18)	

Then, we create a flow balance table. The flow into a node minus the flow out of a node must be zero. In our flow balance table, we have a column for each node with the flow in and a second column with the flow out. The third column is the difference between the two and finally we need to create a column of zeros since that is what the difference should equal.

Flow Balance					
	Flow In	Flow Out	Difference		
Node S	=D19	=(J13)	=(D23-E23)	=	0
Node A	=(E19)	=(J14)	=(D24-E24)	=	0
Node B	=(F19)	=(J15)	=(D25-E25)	=	0
Node C	=(G19)	=(J16)	=(D26-E26)	=	0
Node D	=(H19)	=(J17)	=(D27-E27)	=	0
Node T	=(I19)	=(J18)	=(D28-E28)	=	0

Before we begin to use the solver, we need to set out objective function. The objective function is the maximum flow and will be equal to the value of the dummy arc, or the flow from t to s.

After clicking on the cell with the objective function, we pull up the solver. We input that our second table will be the variable cells. We need to put the two constraints on the variable. The first is that the variables in the table must be less than or equal to the value in the first table. The second constraint is that all of flow differences must be equal to zero. Select the Simplex LP solving method and click on solve.



The results below show us that the maximum flow from s to t is 2300 vehicles per hour and we can also see the flows into and out of each node.

								Total Out	
		Node S	Node A	Node B	Node C	Node D	Node T		
	Node S	0	12	0	11	0	0	23	
	Node A	0	0	12	0	0	0	12	
	Node B	0	0	0	0	0	19	19	
	Node C	0	0	0	0	11	0	11	
	Node D	0	0	7	0	0	4	11	
	Node T	23	0	0	0	0	0	23	
Total In		23	12	19	11	11	23		
	Flow Balance								
		Flow In	Flow Out	Difference					
	Node S	23	23	0	=	0			
	Node A	12	12	0	=	0			
	Node B	19	19	0	=	0			
	Node C	11	11	0	=	0			
	Node D	11	11	0	=	0		Maximum Flow	23
	Node T	23	23	0	=	0			

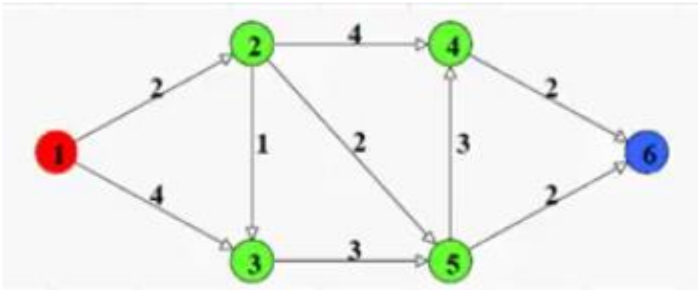
If you need more information regarding the maximum flow using Excel, visit https://www.youtube.com/watch?v=cztF4L370M8 .

Shortest-Path Model

The shortest-path model is used to find the shortest route from one location to another within a network or possibly the cheapest route from one location another.

The shortest-path model has one designated starting node and one designated final destination node. Each of the other nodes are transshipment nodes because products neither

Let's look at example and examine it from a practical viewpoint and then try it with the aid of Microsoft Excel.



Above, we have a six node network with the distance between nodes shown on the arcs. In this example, we are trying to find the shortest route from node 1 to node 6. This is a very simple example and we do not even have to use excel to solve it.

What are the possible routes from node 1 to node 6 and what is the length of the path?

First Arc	Second Arc	Third Arc	Fourth Arc	Fifth Arc	Total Distance
1-2	2-4	4-6			8
1-2	2-3	3-5	5-4	4-6	11
1-2	2-3	3-5	5-6		8
1-2	2-5	5-4	4-6		9
1-2	2-5	5-6			6
1-3	3-5	5-4	4-6		12
1-3	3-5	5-6			9

So the shortest path is 1-2-5-6 and it has a length of 6.

Now, let's look at how to solve this problem using the Excel solver.

First we make a table with the distances from one node to another. For the distance between nodes that are not connected, use a large number to assure that the shortest path does not use this router. For our table, we used 1000 for the distance between unconnected nodes.

	A	B	C	D	E	F	G	H	I
2					To				
3			1	2	3	4	5	6	
4	From	1	1000	2	4	1000	1000	1000	
5		2	1000	1000	1	4	2	1000	
6		3	1000	1000	1000	1000	3	1000	
7		4	1000	1000	1000	1000	1000	2	
8		5	1000	1000	1000	3	1000	2	
9		6	1000	1000	1000	1000	1000	1000	

Now we create a variable table, with sum of each row and column on the last column and last row.

				To				
		1	2	3	4	5	6	Total Out
From	1							0
	2							0
	3							0
	4							0
	5							0
	6							0
Total In		0	0	0	0	0	0	

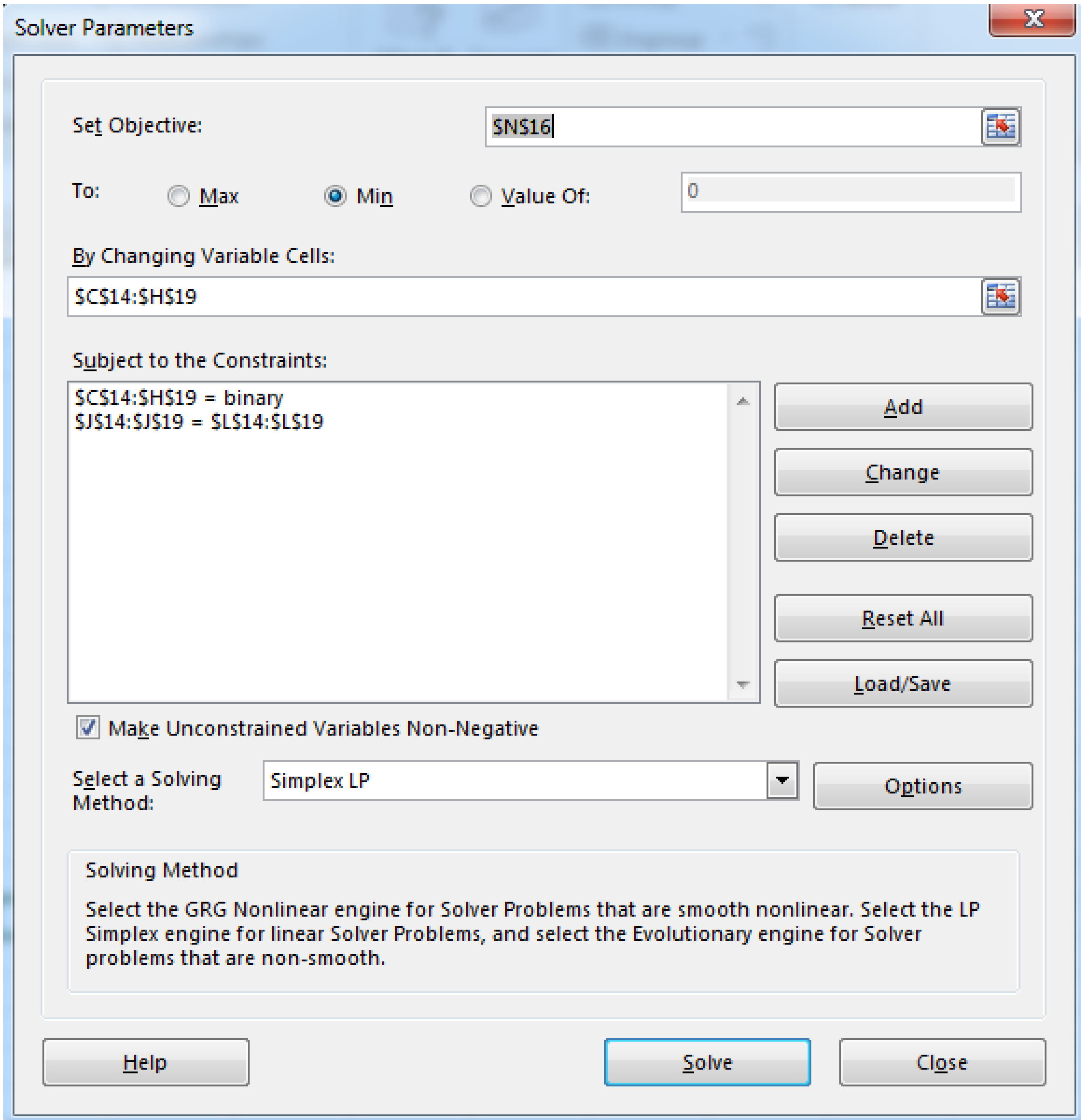
Now we make a separate column with the net flow or the total out minus the total in for each of the nodes.

The net flow of nodes 2, 3, 4 and 5 must be zero, while node one must have a net flow of 1 and node 6 has a net flow of -1.

The objective function is the total distance traveled which is the sumproduct of the two tables.

	A	B	C	D	E	F	G	H	I	J	K
2					To						
3			1	2	3	4	5	6			
4	From	1	1000	2	4	1000	1000	1000		Total Distance	=SUMPRODUCT(C4:H9,C14:H1
5		2	1000	1000	1	4	2	1000			
6		3	1000	1000	1000	1000	3	1000			
7		4	1000	1000	1000	1000	1000	2			
8		5	1000	1000	1000	3	1000	2			
9		6	1000	1000	1000	1000	1000	1000			
10											
11											
12					To						
13			1	2	3	4	5	6	Total Out	LMS	
14	From	1							=SUM(C14:H14)	=I14-C20	=
15		2							=SUM(C15:H15)	=I15-D20	=
16		3							=SUM(C16:H16)	=I16-E20	=
17		4							=SUM(C17:H17)	=I17-F20	=
18		5							=SUM(C18:H18)	=I18-G20	=
19		6							=SUM(C19:H19)	=I19-H20	=
20	Total In		=SUM(C14:C19)	=SUM(D14:D19)	=SUM(E14:E19)	=SUM(F14:F19)	=SUM(G14:G19)	=SUM(H14:H19)			
21											

Click on the objective function cell and then click on the solver. Set the variables to be the cells in the second table. There are two constraints, the first is that the variables are binary since they can only be 1 or 0. The second is that the net flow must be equal to the values that we stated.



Make sure that the program is set to minimize and that the solving method is simplex LP.

After running the program, we can see that Excel's solution is the same as the one we determined by hand.

	A	B	C	D	E	F	G	H	I	J	K	L	M
2					To								
3			1	2	3	4	5	6					
4	From	1	1000	2	4	1000	1000	1000		Total Distance	6		
5		2	1000	1000	1	4	2	1000					
6		3	1000	1000	1000	1000	3	1000					
7		4	1000	1000	1000	1000	1000	2					
8		5	1000	1000	1000	3	1000	2					
9		6	1000	1000	1000	1000	1000	1000					
10													
11													
12					To								
13			1	2	3	4	5	6	Total Out	LMS			
14	From	1	0	1	0	0	0	0	1	1	=	1	
15		2	0	0	0	0	1	0	1	0	=	0	
16		3	0	0	0	0	0	0	0	0	=	0	
17		4	0	0	0	0	0	0	0	0	=	0	
18		5	0	0	0	0	0	1	1	0	=	0	
19		6	0	0	0	0	0	0	0	-1	=	-1	
20	Total In		0	1	0	0	1	1					
21													

For more information regarding the shortest-path model, please visit https://www.youtube.com/watch?v=EUYgMCQwrlY .